Recall:

We insert the ansatz $y = e^{rx}$ into the differential equation

$$ay'' + by' + cy = 0 \qquad (1)$$

and obtain the following constraint:

$$ar^2 + br + c = 0 \quad \Rightarrow r_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\rightarrow$ distinguish 3 cases:

1): $b^2 - 4ac > 0$ ✓

2): $b^2 - 4ac = 0$ 3): $b^2 - 4ac < 0$

Case 2: $b^2 - 4ac = 0$

In this case $r_1 = r_2 \rightarrow$ roots of the characteristic equation are real and equal.

Let's denote by r the common value of r_1 and r_2. Then we have $r = -\frac{b}{2a}$ so $2ar + b = 0$ (2)

We know that $y_1 = e^{rx}$ is a solution of (1)

We now verify that $y_2 = xe^{rx}$ is also a solution:

$a y_2'' + b y_2' + c y_2$

$= a(2re^{rx} + r^2 x e^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx}$

$= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx}$

$= 0(e^{rx}) + 0 \cdot (xe^{rx}) = 0$

$\longrightarrow$ the most general solution is:

$$y = c_1 e^{rx} + c_2 x e^{rx} \qquad (3)$$

<u>Case 3):</u> $b^2 - 4ac < 0$

In this case the roots r_1 and r_2 are complex numbers. We can write

$$r_1 = \alpha + i\beta, \qquad r_2 = \alpha - i\beta$$

where α and β are real numbers.

(In fact, $\alpha = \frac{-b}{2a}$, $\beta = \sqrt{4ac - b^2}/(2a)$.)

Then, using

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

we can write solutions of the differential equation as

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$= c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= e^{\alpha x}\left[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\right]$$

$$= e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$.

$\longrightarrow$ If the roots of the auxiliary equation $ar^2 + br + c = 0$ are complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of

$$ay'' + by' + cy = 0$$

is

$$y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

<u>Remark 9.4</u> (initial value problem):
Similarly to the case of first order equations, an "initial value problem" for a second order equation consists of finding a solution y of the diff. eq. that also satisfies initial conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

It can be shown that, under suitable conditions, there exists a unique solution to this initial value problem.

Example 9.8:

Solve the initial-value problem

$$y'' + y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution:

The characteristic equation is

$$r^2 + r - 6 = (r-2)(r-3) = 0$$

$\rightarrow$ roots are given by $r = 2, -3$.

Thus the general solution is given by

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

This can be verified directly by substituting into the diff. eq.

Plugging in the initial conditions

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - 3c_2 = 0,$$

we get $c_1 = \frac{3}{5}, \quad c_2 = \frac{2}{5}$.

→ The required solution of the initial value problem is:

$$y = \frac{3}{5} e^{2x} + \frac{2}{5} e^{-3x}.$$

<u>Definition 9.8</u> (non-homogeneous equation):

We would now want to tackle the non-homogeneous equation:

$$ay'' + by' + cy = G(x) \qquad (4)$$

where a, b, and c are constants and $G(x)$ is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \qquad (5)$$

is called the "complementary equation" and plays an important role in the solution of the original non-homogeneous eq. (4).

<u>Proposition 9.4:</u>

The general solution of the non-homogeneous differential equation (4) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of equation (4) and y_c is the general solution of the complementary equation (5).

Proof:

We verify that if y is any solution of equation (4), then $y - y_p$ is a solution of the complementary equation (5)

Indeed

$$a(y - y_p)'' + b(y - y_p)' + c(y - y_p)$$

$$= ay'' - a y_p'' + by' - by_p' + cy - cy_p$$

$$= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p)$$

$$= G(x) - G(x) = 0$$

$\rightarrow$ every solution is of the form

$$y(x) = y_p(x) + y_c(x).$$

$\square$

Example 9.9:

Solve the equation $y'' + y' - 2y = x^2$.

$\rightarrow$ the characteristic equation is

$$r^2 + r - 2 = (r-1)(r+2) = 0$$

with roots $r = 1, -2$. So the solution of the complementary equation is

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Since $G(x) = x^2$ is a polynomial of degree 2, we seek a particular solution of the form

$$y_p(x) = Ax^2 + Bx + C$$

$\longrightarrow$ "method of undetermined coefficients"

Then $y_p' = 2Ax + B$ and $y_p'' = 2A$, so substituting, we have

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$

or $\quad -2Ax^2 + (2A - 2B)x + (2A + B - 2C) = x^2$

Thus

$$-2A = 1, \qquad 2A - 2B = 0, \qquad 2A + B - 2C = 0$$

$\longrightarrow$ solution:

$$A = -\frac{1}{2}, \qquad B = -\frac{1}{2}, \qquad C = -\frac{3}{4}.$$

$\rightarrow$ A particular solution: $y_p(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

and, by Prop. 9.4, the general solution is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-2x} - \frac{1}{2}x^2 - \frac{1x}{2} - \frac{3}{4}$$

$\square$

If $G(x)$ is of the form Ce^{kx}, where C and k are constants, then we take as trial solution a function of the form

$$y_p(x) = Ae^{kx}.$$

Example 9.10:

Solve $y'' + 4y = e^{3x}$

$\longrightarrow$ The characteristic equation is

$$r^2 + 4 = 0$$

with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$.

Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$.

Substituting, gives

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $\quad 13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$.

Thus a particular solution is

$$y_p(x) = \frac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{13}e^{3x}.$$

$\square$

§9.5 Systems of Differential Equations

So far, we have looked at differential equations of one unknown function.

More generally, we can have for example models of population growth with several species:

Example 9.11 (predators and preys):

i) In the absence of predators, we obtain the usual exponential growth of the prey:

$$\frac{dR}{dt} = kR, \quad \text{where } k > 0$$

and $R(t)$ is the number of prey.

ii) In the absence of prey, the predator population would decline:

$$\frac{dW}{dt} = -rW, \quad \text{where } r > 0$$

and $W(t)$ is the predator population.

iii) With both species present, we expect:

$$(1) \quad \frac{dR}{dt} = KR - aRW, \quad \frac{dW}{dt} = -rW + bRW$$

where K, r, a and b are positive constants

<u>Note:</u> the two species encounter each other at a rate $\sim RW$

$\longrightarrow$ "predator-prey" equations

<u>solutions:</u>

There are "equilibrium" solutions when

$$\frac{dR}{dt} = \frac{dW}{dt} = 0 \iff KR - aRW = 0 \text{ and } -rW + bRW = 0$$

$$\iff R = \frac{b}{r}, \quad W = \frac{K}{a}$$

For other cases, we proceed as follows.

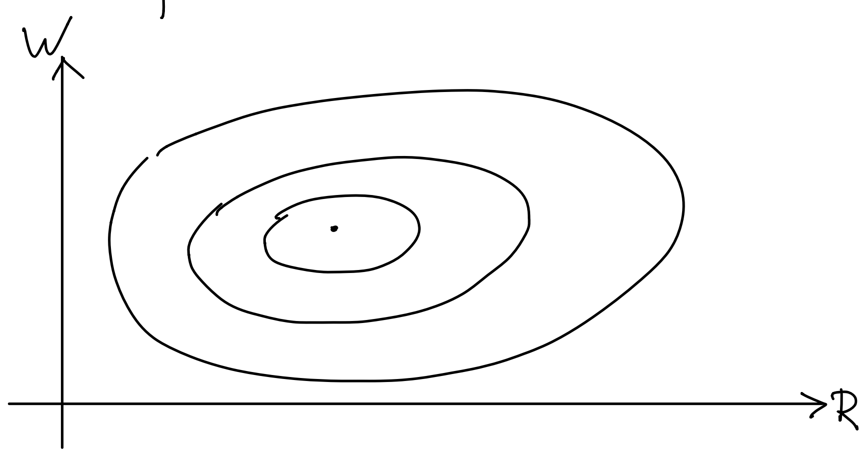
We use the chain rule to eliminate t:

$$\frac{dW}{dt} = \frac{dW}{dR}\,\frac{dR}{dt}$$

so $$\frac{dW}{dR} = \frac{dW/dt}{dR/dt} = \frac{-rW + bRW}{kR - aRW}$$

→ obtained an equation of the form $y' = F(x, y)$

→ can analyse solutions using a direction field method:



We refer to the RW-plane as the "phase plane", and we call the solution curves "phase trajectories".

Equation (1) is an Example of a "system of diff. equations"

Definition 9.9:

i) A system of differential equations is of the form:

$$\frac{dx_1}{dt} = F_1(x_1, x_2, ---, x_n ; t)$$

$$\frac{dx_2}{dt} = F_2(x_1, x_2, ---, x_n ; t) \qquad (2)$$

$$\vdots \qquad \vdots$$

$$\frac{dx_n}{dt} = F_n(x_1, x_2, ---, x_n ; t)$$

where $F_1, F_2, ---, F_n$ are single-valued functions continuous in a certain domain of their arguments, and $x_1, x_2, ---, x_n$ are unknown functions of the real variable t, satisfying (2) for some interval $t \in [t_1, t_2]$.

ii) A solution of (2) is a set of functions $x_1(t), x_2(t), ---, x_n(t)$, called the "components" of $\vec{X}$, and $\vec{X}$ is designated by

$$\vec{X} = (x_1, x_2, ---, x_n)$$

$\longrightarrow$ vector.

If $\vec{X}$ is a function of a real variable t

$$\vec{X}(t) = \{ x_1(t), x_2(t), ---, x_n(t) \}$$

it is "continuous" in t if and only if each of its components is continuous in t.

If each of its components is differentiable, $\vec{X}(t)$ has the "derivative"

$$\frac{d\vec{X}(t)}{dt} = \left\{ \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, ---, \frac{dx_n(t)}{dt} \right\}$$

Thus (2) can be written as

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{x}, t), \qquad (3)$$

where $\vec{F}(\vec{X}, t) = \{ F_1, F_2, ---, F_n \}$.